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Tri-Hamiltonian Structures of The Egorov Systems of Hydrodynamic Type*

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1. Introduction

The study of general Hamiltonian structures for systems of hydrodynamic type, i.e., systems of the form

$$u_t^i = \sum_{k=1}^N v_k^i(u) u_x^k, \quad i = 1, \dots, N, \quad (1)$$

was initiated by Dubrovin and Novikov [2] and continued by Mokhov and Ferapontov [5]. In the present paper, we prove a theorem on the existence of three Hamiltonian structures for a diagonalizable Hamiltonian hydrodynamic type system (1) having two physical symmetries, with respect to the Galilean transformations and to scalings, and possesses some additional properties, namely, the metric of the Hamiltonian structure has the Egorov property and the matrix $((r^i - r^k)\beta_{ik})$ (see Sec. 4 below) is semisimple. There are rather many known physically meaningful systems of hydrodynamic type which have this form, including averaged equations on N -phase solutions of the Korteweg–de Vries equation (the Whitham equations) and the nonlinear Schrödinger equation as well as the dispersionless limits of the vector nonlinear Schrödinger equation and the vector long-short resonance equation. For the Whitham equations, local multi-Hamiltonian structures were described in [15, 1] by the algebro-geometric method. In the present paper, we construct third Hamiltonian structures for all above-mentioned systems by a common differential-geometric method. It turns out that to find out whether there exist second and third Hamiltonian structures and determine their type (local, nonlocal with a constant curvature metric, or general nonlocal), it suffices to know the homogeneity degrees of the annihilators of the first Hamiltonian structure. These degrees are usually *a priori* known. We also prove a simple criterion for the Egorov property of a metric (Theorem 1). Theorems on tri-Hamiltonian structures are stated and proved for the general case in Secs. 2–4 and are then applied to the above-mentioned examples in the subsequent sections.

The theory considered here can be applied not only to integrable diagonalizable systems of hydrodynamic type but also to the theory of Frobenius manifolds and conformal topological field theory for the solution of the Witten–Dijkgraaf–Verlinde–Verlinde associativity equations [18].

Let us briefly state some results needed in the sequel (see [13]).

The original diagonalizable hydrodynamic type system (1) has Riemann invariants $r^i = r^i(u)$, $i = 1, \dots, N$, i.e., variables in which the velocity matrix (v_k^i) of system (1) is diagonal (unless otherwise stated, summation over repeated indices is not assumed):

$$r_t^i = v^i(r) r_x^i, \quad i = 1, \dots, N. \quad (2)$$

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In the following, we assume that $v^i \neq v^k$ for $i \neq k$ and that system (2) *does not split* in the sense that $\partial_i v^k \neq 0$, $i \neq k$. Proceeding from $v^i(r)$, we find the Lamé coefficients $H_i(u)$ as some solutions of the overdetermined system

$$\partial_k \ln H_i = \Gamma_{ik}^i = \frac{\partial_k v^i}{v^k - v^i}, \quad i \neq k. \quad (3)$$

Here $\partial_k \equiv \partial/\partial r^k$ and the Γ_{ik}^i are the Christoffel symbols of the Levi-Civita connection corresponding to the metric entering the Hamiltonian operator \hat{A}^{ij} of the system (see (8) and (9) below). We also find the rotation coefficients (see [13, 17])

$$\beta_{ik} = \frac{\partial_i H_k}{H_i}, \quad i \neq k. \quad (4)$$

The compatibility conditions for systems (3) or (4) have the form of the *semi-Hamiltonian* property [12, 13]

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik} \iff \partial_j \frac{\partial_k v^i}{v^k - v^i} = \partial_k \frac{\partial_j v^i}{v^j - v^i}, \quad i \neq j \neq k. \quad (5)$$

Moreover, the diagonal coefficients of system (2) can be represented in the form $v^i(r) = \overline{H}_i/H_i$, where the H_i are the Lamé coefficients and the \overline{H}_i are some solutions of the system

$$\partial_i \overline{H}_k = \beta_{ik} \overline{H}_i, \quad i \neq k. \quad (6)$$

A local Hamiltonian structure of Dubrovin–Novikov type exists for a semi-Hamiltonian system (2) if and only if

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i, k} \beta_{mi} \beta_{mk} = 0, \quad i \neq k. \quad (7)$$

In this case, the hydrodynamic type system (1) or (2) can be rewritten in the Hamiltonian form

$$u_t^i = \{u^i, H\} = \sum_{j=1}^N \hat{A}^{ij} \frac{\delta H}{\delta u^j} = \sum_{k=1}^N (\nabla^i \nabla_k h) u_x^k \quad (8)$$

with Hamiltonian operator

$$\hat{A}^{ij} = g^{ij}(u) \frac{d}{dx} - \sum_{s,k} g^{is}(u) \Gamma_{sk}^j(u) u_x^k, \quad (9)$$

where $g^{ij} = g^{ji}$ is a nondegenerate zero curvature metric and the connection Γ_{sk}^j is symmetric and compatible with the metric. (Here $H = \int h(u) dx$ is a hydrodynamic type Hamiltonian.) In the Riemann invariants, the metric is diagonal, i.e., $g_{ii} = H_i^2$ ($g^{ii} g_{ii} = 1$), and the vanishing of the Riemann tensor is equivalent to (7). Thus, with each local Hamiltonian structure of system (2), a system of orthogonal curvilinear coordinates r^i in a flat N -dimensional pseudo-Euclidean space is associated. Note that some of the functions H_i and β_{ik} are pure imaginary in the case of pseudo-Riemannian diagonal metrics. (It is metrics of this type that occur in applications, as is elucidated by Theorem 7 below.)

Each coefficient H_i is determined by (3) to within a multiplication by a function of one variable, $H_i = \mu_i^{-1/2}(r^i) H_i^{(\text{fix})}$. In this case, condition (7) acquires the form

$$\frac{1}{2} \mu_i' \beta_{ik}^{(\text{fix})} + \frac{1}{2} \mu_k' \beta_{ki}^{(\text{fix})} + \mu_i \partial_i \beta_{ik}^{(\text{fix})} + \mu_k \partial_k \beta_{ki}^{(\text{fix})} + \sum_{m \neq i, k} \mu_m \beta_{mi}^{(\text{fix})} \beta_{mk}^{(\text{fix})} = 0. \quad (10)$$

Since system (10) is linear in μ_i , we can readily see that any two Hamiltonian structures (9) associated with a given diagonal hydrodynamic type system with nondegenerate metric g_{ij} are automatically compatible. As was shown in [20], system (10) for $\mu_i(r^i)$ can have at most $N+1$ linearly independent solutions. In the general case, it has no solutions at all for a given semi-Hamiltonian system.

The hydrodynamic type system (1) can also have the nonlocal Hamiltonian structure (8) with a nonlocal Hamiltonian operator [11]

$$\hat{A}^{ij} = g^{ij}d/dx - \sum_{s,k} g^{is}\Gamma_{sk}^j u_x^k + \sum_{\alpha,\beta,n,m} \varepsilon_{\alpha\beta} w_n^{i,(\alpha)} u_x^n (d/dx)^{-1} w_m^{j,(\beta)} u_x^m. \quad (11)$$

Here the $w_n^{i,(\alpha)}(u)$ are the matrix coefficients of the hydrodynamic type flows

$$u_{t(\alpha)}^i = \sum_{j=1}^N w_j^{i,(\alpha)}(u) u_x^j \quad (12)$$

commuting with (1). In the Riemann invariants, the coefficients $w_n^{i,(\alpha)}$ are diagonal, namely, $w_i^{i,(\alpha)} = H_i^{(\alpha)}/H_i$, where $H_i^{(\alpha)}$ is a set of solutions of system (6) such that

$$\partial_i \beta_{ik} + \partial_k \beta_{ki} + \sum_{m \neq i,k} \beta_{mi} \beta_{mk} = \sum_{\alpha,\beta} \varepsilon_{\alpha\beta} H_i^{(\alpha)} H_k^{(\beta)}, \quad i \neq k, \quad (13)$$

$\varepsilon_{\alpha\beta} = \varepsilon_{\beta\alpha} = \text{const.}$ Thus, the metric $g^{ij}(u)$ in the above nonlocal Hamiltonian operator is not flat. If Eq. (13) contains a single term $H_i H_k$ formed of the Lamé coefficients $H_i = \sqrt{g_{ii}}$ of the *given metric*, we obtain a constant curvature metric [5].

2. The Egorov Systems of Hydrodynamic Type

As is shown in [12, 13], a diagonal semi-Hamiltonian system (2) has infinitely many commuting hydrodynamic type flows

$$r_y^i = w^i(r) r_x^i, \quad i = 1, \dots, N, \quad (14)$$

whose coefficients $w^i(r)$ can be found as solutions of the consistent overdetermined system of first-order linear partial differential equations

$$\partial_k w^i = \Gamma_{ik}^i (w^k - w^i), \quad i \neq k. \quad (15)$$

System (2) also has infinitely many hydrodynamic conservation laws $dp(r)/dt = dq(r)/dx$, where the conservation law densities $p(r)$ can be found from the consistent system

$$\partial_i \partial_k p = \Gamma_{ik}^i \partial_i p + \Gamma_{ik}^k \partial_k p, \quad i \neq k. \quad (16)$$

The substitution $w^i = \tilde{H}_i/H_i$ reduces system (15) to the form (6), and the substitution $\partial_i p = \psi_i H_i$ reduces (16) to the system

$$\partial_k \psi_i = \beta_{ik} \psi_k, \quad i \neq k, \quad (17)$$

which is the adjoint of (6).

Definition 1. An orthogonal coordinate system associated with the diagonal metric $g_{ii} = H_i^2$ is called a *The Egorov coordinate system* if

$$\beta_{ik} = \beta_{ki}. \quad (18)$$

In this case, the metric is *potential*, i.e., $g_{ii} = \partial_i a(r)$ for some function $a(r)$.

The corresponding hydrodynamic type systems will also be called The Egorov systems. Condition (18) is not invariant with respect to the natural transformation $r^i \rightarrow \varphi^i(r^i)$ of

Riemann invariants or the transformation $H_i \rightarrow \mu_i^{-1/2}(r^i)H_i$. An invariant condition for a nonsplitting system ($\beta_{ik} \neq 0$) with $N \geq 3$ (see [3, 17]) is given by the relation

$$\beta_{ik}\beta_{kj}\beta_{ji} = \beta_{ij}\beta_{jk}\beta_{ki}, \quad i \neq j \neq k. \quad (19)$$

For a Egorov system of hydrodynamic type, the relations $\partial_i p = H_i \bar{H}_i$ hold and systems (6) and (17) coincide. Integrable systems of hydrodynamic type have infinitely many independent conservation laws $\partial_t p_k(u) = \partial_x q_k(u)$, $k = 1, 2, \dots$, and hence can be rewritten in the conservative form

$$\partial_t a^\alpha = \partial_x q^\alpha(a), \quad \alpha = 1, \dots, N, \quad (20)$$

where the $a^\alpha = a^\alpha(u)$ are conservation law densities and the $q^\alpha(a)$ are the corresponding flows.

Let us prove the following *criterion for the Egorov property* of system [8]:

Theorem 1. *A diagonalizable nonsplitting semi-Hamiltonian system (1) has a pair of conservation laws of the form*

$$a_t = b_x, \quad b_t = c_x \quad (21)$$

if and only if it is a Egorov system. In this case, in the Riemann invariants, the relations $\partial_i a = H_i^2$, $\partial_i b = H_i \bar{H}_i$, and $\partial_i c = \bar{H}_i^2$ hold, where $\bar{H}_i = v^i(r)H_i$.

Proof. Condition (21) can be rewritten as

$$\partial_i a = \psi_i H_i, \quad \partial_i b = \psi_i \bar{H}_i, \quad (22)$$

$$\partial_i b = \bar{\psi}_i H_i, \quad \partial_i c = \bar{\psi}_i \bar{H}_i, \quad (23)$$

where the ψ_i and $\bar{\psi}_i$ are solutions of system (17), whence it follows that $\bar{\psi}_i H_i = \psi_i \bar{H}_i$. Differentiating $\bar{\psi}_i = \psi_i \bar{H}_i / H_i$ with respect to the Riemann invariant r^k ($i \neq k$) and dividing the result by $v_i - v_k = \bar{H}_i / H_i - \bar{H}_k / H_k$, we obtain

$$\frac{\psi_k}{H_k} \beta_{ik} = \frac{\psi_i}{H_i} \beta_{ki} \quad (24)$$

from (17), which implies (19) provided that the number N of equations is greater than 2. Note that, by virtue of (3) and (4), the nonsplitting condition ($\partial_i v^k \neq 0$, $i \neq k$) implies that $\beta_{ki} \neq 0$ for $i \neq k$ and by virtue of (24), it guarantees that $\psi_i \neq 0$ and $\partial_i a \neq 0$ for any i . To prove the Egorov property for the case in which system (1) contains only two equations, we set $q_i = \psi_i / H_i = \bar{\psi}_i / \bar{H}_i$. It follows from (23) that $\partial_i(q_k H_k^2) = \partial_k(q_i H_i^2)$ and $\partial_i(q_k \bar{H}_k H_k) = \partial_k(q_i \bar{H}_i H_i)$, whence, using (24), we arrive at the relations $H_i^2 \partial_k q_i = H_k^2 \partial_i q_k$ and $\bar{H}_i H_i \partial_k q_i = \bar{H}_k H_k \partial_i q_k$. Dividing one of these expressions by the other (under the assumption that $\partial_k q_i \neq 0$, $i \neq k$), we obtain $v_i = v_k$, which contradicts the assumption that $v_i \neq v_k$ for $i \neq k$. Consequently, $\partial_k q_i = 0$. Since the Riemann invariants are defined only to within the transformation $r^i \rightarrow \varphi^i(r^i)$, we can always make the functions $q_i(r^i) = \psi_i / H_i$ in (24) identically equal to unity. Thus, the metric is potential and we have $\partial_i a = H_i^2$, $\partial_i b = H_i \bar{H}_i$, and $\partial_i c = \bar{H}_i^2$.

Conversely, if the metric is potential, i.e., $H_i^2 = \partial_i a$, then conditions (18) hold. In this case, we have $a_t = \partial_i a \cdot r_t^i = H_i^2 r_t^i = H_i^2 v^i(r) r_x^i = H_i \bar{H}_i r_x^i = b_x$ and $b_t = \partial_i b \cdot r_t^i = H_i \bar{H}_i r_t^i = H_i \bar{H}_i v^i(r) r_x^i = \bar{H}_i^2 r_x^i = c_x$. ■

Note that the condition that the metric is flat has not been used in the proof of the theorem. Egorov himself studied potential metrics with the additional zero curvature condition. In this paper, potential metrics of arbitrary curvature will be referred to as *Egorov metrics*.

Since all commuting flows have the same metric (see (3) and (15)), we arrive at the following result.

Corollary 1. *An arbitrary commuting flow (14) of a Egorov semi-Hamiltonian system of hydrodynamic type has a pair of conservation laws of the form (21) $\partial_y a = \partial_x h$, $\partial_y h = \partial_x g$.*

It is obvious that if the hydrodynamic type system (2) has the local Hamiltonian structure (8), then the relationship $w_k^i(u) = \nabla^i \nabla_k p$ between the solutions of Eqs. (16) for conservation law densities and the solutions of Eqs. (15) for commuting flows holds. Writing the operation of covariant differentiation ∇^i in full and substituting the expressions $w^i(r) = \overline{H}_i/H_i$ and $\partial_i p = \psi_i H_i$ (in Riemann invariants), we obtain a relation between the solutions of problem (6) and the adjoint problem (17),

$$\overline{H}_i = \partial_i \psi_i + \sum_{m \neq i} \beta_{mi} \psi_m. \quad (25)$$

If a hydrodynamic type system has a second local Hamiltonian structure, then, as was already noted, the diagonal coefficients $g_{(2)}^{ii}$ of its metric can differ only in factors depending on the corresponding Riemann invariants, $g_{(2)}^{ii} = \mu_i(r^i) g_{(1)}^{ii}$. Let us make a transformation $r^i \rightarrow \varphi^i(r^i)$ of the Riemann invariants that gives $\mu_i(r^i) \rightarrow r^i$ (under the assumption that $\mu'_i(r^i) \neq 0$), i.e., $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$. By analogy with (25), for the connection generated by the second flat metric $g_{(2)}^{ii}$ we obtain

$$\tilde{H}_i = \frac{1}{2} \psi_i + r^i \partial_i \psi_i + \sum_{m \neq i} r^m \beta_{mi} \psi_m. \quad (26)$$

If $g_{(1)}^{ii}$ is an Egorov metric, then formulas (25) and (26) imply that

$$\overline{H}_i = \delta \psi_i, \quad \tilde{H}_i = (\hat{R} + 1/2) \psi_i, \quad (27)$$

where $\delta = \sum \partial_k$ is a translation operator and $\hat{R} = \sum r^k \partial_k$ is a scaling operator (see [10]).

Let us summarize the preceding. For an appropriate choice of Riemann invariants, if an Egorov hydrodynamic type system has one local Hamiltonian structure, then $\delta \beta_{ik} = 0$; in other words, the rotation coefficients of orthogonal coordinate systems depend only on the differences of Riemann invariants, $\beta_{ik} = \beta_{ik}(r^m - r^n)$. If the Egorov hydrodynamic type system has also a second local Hamiltonian structure with metric $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$, then Eqs. (10) and (5) imply the equivalent homogeneity condition $\hat{R} \beta_{ik} = -\beta_{ik}$; thus, the rotation coefficients of orthogonal coordinate systems are homogeneous functions of order -1 . As is shown in [13], the Egorov property of the metric for a diagonalizable Hamiltonian system essentially expresses the fact that the system is invariant with respect to the group of Galilean transformations $(x, t) \rightarrow (x - V \cdot t, t)$. The homogeneity of rotation coefficients in some well-known examples corresponds to the natural physical condition of invariance with respect to the choice of units of measurement, which results in the homogeneity of $v_i(r)$. Thus, the presence of two local Hamiltonian structures for physical examples of hydrodynamic type systems is a manifestation of the simplest fundamental invariance conditions. We have already noted that the corresponding Hamiltonian structures prove to be compatible automatically.

In the following, for brevity we say “homogeneous rotation coefficients”, implying that the degree of homogeneity is always -1 , $\hat{R} \beta_{ik} = -\beta_{ik}$. The corresponding curvilinear orthogonal coordinate systems and diagonal hydrodynamic type systems (2) will also be simply said to be homogeneous.

3. Annihilators of the First Local Hamiltonian Structure and the Signature of its Metric for the Homogeneous Egorov Systems

Annihilators of a local Hamiltonian structure are defined as conservation law densities $a^\alpha(u)$ determined by the vanishing condition for commuting flows:

$$0 = \nabla^i \nabla_k a^\alpha, \quad \alpha = 1, \dots, N. \quad (28)$$

The overdetermined system (28) has an N -dimensional solution space, which specifies flat coordinates for the metric $g_{ii} = H_i^2$ of the Hamiltonian structure. Introducing $\psi_i^{(\alpha)} = \partial_i a^\alpha / H_i$ in the Riemann invariants, we rewrite Eq. (28) in the form

$$\partial_k \psi_i^{(\alpha)} = \beta_{ik} \psi_k^{(\alpha)}, \quad 0 = \partial_i \psi_i^{(\alpha)} + \sum_{m \neq i} \beta_{mi} \psi_m^{(\alpha)}, \quad \alpha = 1, \dots, N. \quad (29)$$

The following two theorems were proved in [14] and later presented in [18].

Theorem 2. *The homogeneous solutions $\psi_i^{(\alpha)}$ of system (29) for an Egorov orthogonal coordinate system whose rotation coefficients are homogeneous and depend only on the difference of the Riemann invariants are eigenvectors of the skew-symmetric matrix \mathbf{B} with entries $B_{ij} = (r^i - r^j) \beta_{ij}$, which has constant eigenvalues.*

Proof. We have

$$\begin{aligned} \widehat{R} \psi_i^{(\alpha)} &= r^i \psi_{i,i}^{(\alpha)} + \sum_{m \neq i} r^m \partial_m \psi_i^{(\alpha)} \\ &= \sum_{m \neq i} r^m \beta_{mi} \psi_m^{(\alpha)} - r^i \sum_{m \neq i} \beta_{mi} \psi_m^{(\alpha)} = \sum_{m \neq i} (r^m - r^i) \beta_{mi} \psi_m^{(\alpha)} = c_\alpha \psi_i^{(\alpha)}. \end{aligned}$$

The constancy of the eigenvalues $c_\alpha(r)$ of the matrix \mathbf{B} can be proved as follows: $\partial_i (\widehat{R} \psi_k^{(\alpha)}) = \widehat{R} (\partial_i \psi_k^{(\alpha)}) + \partial_i \psi_k^{(\alpha)} = \widehat{R} (\beta_{ki} \psi_i^{(\alpha)}) + \beta_{ki} \psi_i^{(\alpha)} = \widehat{R} (\beta_{ki}) \psi_i^{(\alpha)} + \beta_{ki} \widehat{R} (\psi_i^{(\alpha)}) + \beta_{ki} \psi_i^{(\alpha)} = -\beta_{ki} \psi_i^{(\alpha)} + c_\alpha \psi_i^{(\alpha)} \beta_{ki} + \beta_{ki} \psi_i^{(\alpha)} = c_\alpha \psi_i^{(\alpha)} \beta_{ki}$, i.e., $\partial_i (c_\alpha \psi_k^{(\alpha)}) = \partial_i (c_\alpha) \psi_k^{(\alpha)} + c_\alpha \beta_{ki} \psi_i^{(\alpha)} = c_\alpha \psi_i^{(\alpha)} \beta_{ki}$, whence $\partial_i (c_\alpha) \equiv 0$. ■

The expressions $\psi_i^{(\alpha)}$ will also be referred to as *annihilators*.

Since the homogeneity degrees of any conservation law densities, including annihilators, are real numbers in the case of homogeneous real hyperbolic systems (2) considered here, it follows that the eigenvalues of the *skew-symmetric* matrix \mathbf{B} are also real. This can be accounted for by the fact that for this class of hydrodynamic type systems the metric is necessarily pseudo-Euclidean (see Theorem 7 below), i.e., some Lamé coefficients $H_i = \sqrt{g_{ii}}$ and rotation coefficients β_{ik} are pure imaginary. Thus, the matrix \mathbf{B} has both real and pure imaginary entries, and one can no longer claim that there is a complete set of eigenvectors. For example, if $N = 3$, then for an Egorov homogeneous metric of signature 1 one can choose coefficients β_{ij} such that $(r^1 - r^2)^2 \beta_{12}^2 + (r^1 - r^3)^2 \beta_{13}^2 + (r^2 - r^3)^2 \beta_{23}^2 = 0$. (It is easily seen that this condition is compatible with Eqs. (30) for β_{ik} , given below.) In this situation, the matrix \mathbf{B} has one eigenvector and two root vectors with eigenvalue 0. Note that the set of The Egorov orthogonal coordinate systems of given dimension with homogeneous rotation coefficients depending only on the differences of Riemann invariants is a finite-parameter family, and the description of such systems can be reduced to that of solutions of the consistent Pfaff system

$$\partial_i \beta_{jk} = \beta_{ji} \beta_{ik}, \quad \delta \beta_{ik} = 0, \quad \widehat{R} \beta_{ik} = -\beta_{ik}, \quad i \neq j \neq k, \quad (30)$$

whose solution can be specified by setting an *arbitrary* symmetric matrix (β_{ik}) (where the β_{ii} are undefined) at the initial point $(r_{(0)}^i)$.

However, all physical examples of diagonalizable homogeneous Egorov systems known to the authors, as well as The Egorov orthogonal coordinate systems arising in conformal topological field theory, for which the rotation coefficients are homogeneous and depend only on the differences of Riemann invariants, result in matrices \mathbf{B} with *complete sets of eigenvectors*. Moreover, as a rule, the homogeneity degrees of annihilators of the first local Poisson bracket are *a priori* known for physical systems. Hence we assume throughout the following that the *semisimplicity condition* holds, namely, the matrix \mathbf{B} has a complete set of eigenvectors at some point $(r_{(0)}^i)$. As was shown in [18], the description of semisimple solutions of system (30) can be reduced to the solution of the sixth Painlevé equation.

Theorem 3. *For a semisimple matrix \mathbf{B} , all annihilators $\psi_i^{(\alpha)}$ corresponding to a flat homogeneous Egorov coordinate system can be taken in the form of homogeneous functions,*

$$\mathbf{B}\vec{\psi}^{(\alpha)} = \widehat{R}\vec{\psi}^{(\alpha)} = c_\alpha \vec{\psi}^{(\alpha)}, \quad \alpha = 1, \dots, N, \quad (31)$$

where the c_α (the homogeneity degrees of the annihilators) are simultaneously the eigenvalues of the matrix \mathbf{B} .

Proof. For a symmetric matrix (β_{ik}) , i.e., for The Egorov metrics, the second relation in (29) becomes $\delta\psi_i^{(\alpha)} = \partial_i\psi_i^{(\alpha)} + \sum_{m \neq i} \beta_{im}\psi_m^{(\alpha)} = \partial_i\psi_i^{(\alpha)} + \sum_{m \neq i} \partial_m\psi_i^{(\alpha)} = 0$. Thus, the (possibly nonhomogeneous) functions $\psi_i^{(\alpha)}$ satisfy the system

$$\delta\psi_i^{(\alpha)} = 0, \quad \partial_i\psi_k^{(\alpha)} = \beta_{ik}\psi_i^{(\alpha)}, \quad i \neq k. \quad (32)$$

One can readily verify that this system is consistent and the initial data for it are given by N constants, namely, the values of the function $\vec{\psi}^{(\alpha)}$ at some point $(r_{(0)}^i)$. Reproducing the computations made in the proof of the preceding theorem, we see that $\partial_i(\widehat{R}\vec{\psi}_k - c\vec{\psi}_k) = \beta_{ik}(\widehat{R}\vec{\psi}_i - c\vec{\psi}_i)$ and $\delta(\widehat{R}\vec{\psi}_k - c\vec{\psi}_k) = 0$ for each solution $\vec{\psi}_i$ and each $c = \text{const}$. Thus, $\vec{\psi}_i = \widehat{R}\vec{\psi}_i - c\vec{\psi}_i$ also satisfies system (32). Consequently, equipping system (32) with the initial data $\vec{\psi}^{(\alpha)}(r_{(0)}^i)$ forming a basis of eigenvectors of the matrix \mathbf{B} at this point $(r_{(0)}^i)$, we see that the $\vec{\psi}_i^{(\alpha)} = \widehat{R}\vec{\psi}_i^{(\alpha)} - c_\alpha\vec{\psi}_i^{(\alpha)}$ are also solutions of system (32) that vanish at the point $(r_{(0)}^i)$. Thus, $\vec{\psi}_i^{(\alpha)} \equiv 0$, as desired. ■

Definition 2. The *momentum density* P for a system of hydrodynamic type with local Hamiltonian structure is understood as the quadratic expression

$$P = \frac{1}{2} \sum_{\alpha, \beta} g_{\alpha\beta} a^\alpha a^\beta, \quad (33)$$

where the nonsingular matrix $g_{\alpha\beta}$ is the (constant) metric of system (2) in flat annihilator coordinates a^α .

It is obvious (see [2]) that in the annihilator coordinates a^α the hydrodynamic type system (1) has the form

$$a_t^\alpha = \frac{d}{dx} \left(\sum_{\beta=1}^N g^{\alpha\beta} \frac{\partial h}{\partial a^\beta} \right), \quad (34)$$

where $h(u)$ is the Hamiltonian density. The momentum density gives an integral of an arbitrary hydrodynamic type Hamiltonian system (34) with given metric $g^{\alpha\beta}$ and generates the trivial commuting flow $u_t^i = u_x^i$.

Note the following assertion (proved independently in several papers by different authors).

Theorem 4. *If a (not necessarily diagonalizable) hydrodynamic type system (1) has $N+1$ linearly independent hydrodynamic conservation laws and one of their densities can be expressed quadratically via the others (see (33)), then the system has the local Hamiltonian structure (34) with constant metric $g_{\alpha\beta}$.*

Proof. Let $\partial_t a^\alpha = \partial_x q^\alpha$ and $\partial_t P = \partial_t [\frac{1}{2} \sum_{\alpha,\beta} g_{\alpha\beta} a^\alpha a^\beta] = \partial_x Q(a)$. It follows that

$$dQ = \sum_{\alpha,\beta} g_{\alpha\beta} a^\alpha dq^\beta(a) = \sum_{\alpha,\beta} d[g_{\alpha\beta} a^\alpha q^\beta] - \sum_{\alpha,\beta} g_{\alpha\beta} q^\beta da^\alpha.$$

Consequently, $\sum_\beta g_{\alpha\beta} q^\beta \equiv \partial h / \partial a^\alpha$ for some function $h(u)$, and hence system (20) acquires the form (34). ■

The following theorem, generalizing Theorem 4 to the case of a constant curvature metric, was stated in [7] and proved in [25].

Theorem 5. *Suppose that a hydrodynamic type system represented in the form of conservation laws $\partial_t c^\alpha = \partial_x b^\alpha$ has an additional conservation law density p quadratically related with the densities c^α :*

$$p - \frac{\varepsilon}{2} p^2 = \frac{1}{2} \sum_{\alpha,\beta} \bar{g}_{\alpha\beta} c^\alpha c^\beta, \quad (35)$$

where $\bar{g}_{\alpha\beta}$ is a constant nonsingular matrix. Then in the field variables c^α the system has the nonlocal Hamiltonian structure

$$c_t^\alpha = \partial_x \left[\sum_{\beta=1}^N (\bar{g}^{\alpha\beta} - \varepsilon c^\alpha c^\beta) \frac{\partial h}{\partial c^\beta} + \varepsilon c^\alpha h \right] \quad (36)$$

associated with the metric $g^{\alpha\beta} = \bar{g}^{\alpha\beta} - \varepsilon c^\alpha c^\beta$ of constant curvature ε , where $(\bar{g}^{\alpha\beta})$ is the inverse matrix of $(\bar{g}_{\alpha\beta})$.

Proof. It follows from the formulas $\partial_t c^\alpha = \partial_x b^\alpha$ and $p_t = \partial_x Q(c)$ and from (35) that

$$dQ = \sum_{\alpha,\beta} \bar{g}_{\alpha\beta} \frac{c^\beta}{1 - \varepsilon p} db^\alpha.$$

By setting $q^\alpha = c^\alpha / (1 - \varepsilon p)$, we find that $\sum_\beta \bar{g}_{\alpha\beta} b^\beta = \partial s / \partial q^\alpha$ with some potential s . Hence we see that

$$p_t = \partial_x \left[\sum_\beta q^\beta \frac{\partial s}{\partial q^\beta} - s \right] \quad \text{and} \quad c_t^\alpha = \partial_x \left[\sum_\beta \bar{g}^{\alpha\beta} \frac{\partial s}{\partial q^\beta} \right].$$

Since

$$\frac{\partial s}{\partial q^\alpha} = (1 - \varepsilon p) \left[\frac{\partial s}{\partial c^\alpha} - \sum_{\beta,\gamma} \varepsilon c^\gamma \frac{\partial s}{\partial c^\gamma} \bar{g}_{\alpha\beta} c^\beta \right],$$

we readily obtain (36) with $h = (1 - \varepsilon p)s$. ■

Needless to say, the shift $p \rightarrow p + \text{const}$ makes relation (35) purely quadratic. However, the form given in (35) clarifies the passage to the limit as $\varepsilon \rightarrow 0$.

Theorem 6. *The following relations hold for the components of an arbitrary (not necessarily Egorov or homogeneous) flat metric in flat annihilator coordinates:*

$$\sum_{\alpha,\beta} g_{\alpha\beta} \psi_i^{(\alpha)} \psi_k^{(\beta)} = \delta_{ik}, \quad g^{\alpha\beta} = \sum_{i=1}^N \psi_i^{(\alpha)} \psi_i^{(\beta)}. \quad (37)$$

For the special case of an Egorov metric, this result was obtained in [19].

Proof. The momentum density P (see (33)) satisfies the relation

$$\partial_i P = \Psi_i H_i = \sum_{\alpha,\beta} g_{\alpha\beta} a^\beta \psi_i^{(\alpha)} H_i \implies \Psi_i = \sum_{\alpha,\beta} g_{\alpha\beta} a^\beta \psi_i^{(\alpha)}.$$

Differentiating it with respect to r^k and using (17), we obtain $\sum_{\alpha,\beta} g_{\alpha\beta} \psi_i^{(\alpha)} \psi_k^{(\beta)}|_{i \neq k} = 0$. Taking account of the fact that P generates the flow $u_t^i = u_x^i$ and differentiating with respect to r^i , we obtain $\sum_{\alpha,\beta} g_{\alpha\beta} \psi_i^{(\alpha)} \psi_i^{(\beta)} = 1$, which precisely gives the first formula in (37). Multiplying it by $\psi_k^{(\gamma)}$ and performing summation over k , we arrive at the relation

$$\sum_{k=1}^N \left(\sum_{\alpha,\beta} g_{\alpha\beta} \psi_i^{(\alpha)} \psi_k^{(\beta)} \right) \psi_k^{(\gamma)} = \sum_{\alpha=1}^N \psi_i^{(\alpha)} \left(\sum_{\beta=1}^N g_{\alpha\beta} \sum_{k=1}^N \psi_k^{(\beta)} \psi_k^{(\gamma)} \right) = \psi_i^{(\gamma)}.$$

Since the vectors $\vec{\psi}^{(\alpha)}$ are linearly independent, it follows that

$$\sum_{\beta=1}^N g_{\alpha\beta} \left(\sum_{k=1}^N \psi_k^{(\beta)} \psi_k^{(\gamma)} \right) = \delta_\alpha^\gamma,$$

i.e., we have arrived at the second formula in (37). ■

Let us return to the case of an Egorov orthogonal coordinate system whose rotation coefficients are homogeneous and depend only on the differences of Riemann invariants. Then

$$\widehat{R} g^{\alpha\beta} = 0 = \widehat{R} \sum_{i=1}^N \psi_i^{(\alpha)} \psi_i^{(\beta)} = \sum_{i=1}^N \psi_i^{(\alpha)} (\widehat{R} \psi_i^{(\beta)}) + \sum_{i=1}^N \psi_i^{(\beta)} (\widehat{R} \psi_i^{(\alpha)}) = (c_\alpha + c_\beta) \sum_{i=1}^N \psi_i^{(\alpha)} \psi_i^{(\beta)}.$$

It follows that either $c_\alpha + c_\beta \neq 0$, and then the corresponding component of the matrix $(g^{\alpha\beta})$ is zero, or $c_\alpha + c_\beta = 0$, and then the corresponding component of the matrix $(g^{\alpha\beta})$ is nonzero. (Some of the components can also be zero, but there must be sufficiently many nonzero components to ensure that the metric is nondegenerate.) Hence if the semisimple matrix \mathbf{B} has at most one eigenvector with zero eigenvalue, then, after an appropriate renumbering of the flat coordinates corresponding to the homogeneous functions $\psi_i^{(\alpha)}$, the flat metric acquires the block antidiagonal form

$$g^{\alpha\beta} = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix}, \quad g^{\alpha\beta} = \begin{pmatrix} 0 & 0 & * \\ \vdots & 0 & \\ 0 & \pm 1 & 0 & \dots & 0 \\ * & 0 & 0 & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}. \quad (38)$$

$N = 2n \qquad N = 2n + 1$

Thus we have proved the following assertion.

Theorem 7. *If the multiplicity of the zero eigenvalue of the semisimple matrix \mathbf{B} does not exceed 1 for a flat homogeneous Egorov coordinate system, then the signature of the given metric $g^{\alpha\beta}$ (the difference between the numbers of positive and negative diagonal coefficients in a diagonal form of the metric) is minimal. (It is equal to 0 for N even and to ± 1 for N odd.)*

4. The third Hamiltonian structure of the Egorov homogeneous systems

As was noted in Sec. 2, our physical examples of hydrodynamic type systems have two local Hamiltonian structures owing to the homogeneity and the invariance with respect to Galilean transformations.

It is known from linear algebra that a skew-symmetric matrix \mathbf{B} has pairs of eigenvalues differing in sign. In what follows, we sometimes do not distinguish between eigenvalues c_α of \mathbf{B} differing in sign.

Let us expand the square of \mathbf{B} as follows:

$$\begin{aligned} [\mathbf{B}^2]_{ik} &= \sum_{m \neq i, k} (r^m - r^i)(r^m - r^k) \beta_{mi} \beta_{mk} \\ &= \left[\sum_{m \neq i, k} (r^m)^2 \beta_{mi} \beta_{mk} + (r^i)^2 \partial_i \beta_{ik} + (r^k)^2 \partial_k \beta_{ki} + r^i \beta_{ik} + r^k \beta_{ki} \right] \\ &\quad - (r^i + r^k) \left[\sum_{m \neq i, k} r^m \beta_{mi} \beta_{mk} + r^i \partial_i \beta_{ik} + r^k \partial_k \beta_{ki} + \frac{1}{2} \beta_{ik} + \frac{1}{2} \beta_{ki} \right] \\ &\quad + r^i r^k \left[\sum_{m \neq i, k} \beta_{mi} \beta_{mk} + \partial_i \beta_{ik} + \partial_k \beta_{ki} \right] + \frac{1}{2} (r^i - r^k) (\beta_{ki} - \beta_{ik}). \end{aligned} \quad (39)$$

Theorem 8. *For even $N = 2n$, a homogeneous Egorov diagonal hydrodynamic type system (2) with flat metric $g_{(1)}^{ii}$ and second flat metric $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$ has a third local Hamiltonian structure with metric $g_{(3)}^{ii} = (r^i)^2 g_{(1)}^{ii}$ if and only if the matrix \mathbf{B} is nonsingular and semisimple and has only one pair of eigenvalues $\pm c$.*

Proof. In this case, $\mathbf{B}^2 = c^2 E$, where E is the identity matrix, since an application of \mathbf{B}^2 to any of its eigenvectors $\psi_i^{(\alpha)}$ gives $c^2 \psi_i^{(\alpha)}$. On the other hand, it follows from (39) that in the presence of two local Hamiltonian structures determined by metrics $g_{(1)}^{ii}$ and $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$ (cf. (10) with $\mu_i = r^i$), one has

$$[\mathbf{B}^2]_{ik} = \sum_{m \neq i, k} (r^m)^2 \beta_{mi} \beta_{mk} + (r^i)^2 \partial_i \beta_{ik} + (r^k)^2 \partial_k \beta_{ki} + r^i \beta_{ik} + r^k \beta_{ki} = 0$$

provided that $i \neq k$, which precisely implies the existence of a third local Hamiltonian structure (cf. (10) with $\mu_i = (r^i)^2$).

Conversely, if condition (10) with $\mu_i = (r^i)^2$ holds, then the matrix \mathbf{B}^2 is diagonal. Since system (2) does not split, it follows that $\beta_{ik} \neq 0$ for $i \neq k$. By (17), it is obvious that none of the components of any homogeneous annihilator $\psi_i^{(\alpha)}$ vanishes identically. Since $\mathbf{B}^2 \psi_i^{(\alpha)} = c_\alpha^2 \psi_i^{(\alpha)}$, we see that the diagonal matrix \mathbf{B}^2 has identical diagonal entries c_α^2 . One can readily prove that the matrix \mathbf{B} is semisimple by applying \mathbf{B}^2 to root vectors; namely, if $\mathbf{B} \vec{v} = c_\alpha \vec{v} + \vec{\psi}^{(\alpha)}$, then $\mathbf{B}^2 \vec{v} = c_\alpha^2 \vec{v} + 2c_\alpha \vec{\psi}^{(\alpha)} \neq c_\alpha^2 \vec{v}$. ■

Theorem 9. *For odd $N = 2n + 1$, a homogeneous Egorov diagonal hydrodynamic type system (2) with flat metric $g_{(1)}^{ii}$ and second flat metric $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$ has a third nonlocal*

Hamiltonian structure with constant curvature metric $g_{(3)}^{ii} = (r^i)^2 g_{(1)}^{ii}$ if and only if the matrix \mathbf{B} is semisimple and has a simple eigenvalue $c_{(0)} = 0$ and a single pair of nonzero eigenvalues $c_{(\pm k)} = \pm c$ ($k = 1, \dots, n$) and the metric $g_{(1)}^{ii}$ itself is homogeneous of degree 0 and depends only on the differences of the Riemann invariants r^i .

Proof. In this case, $[\mathbf{B}^2]_{ij} = c^2 \delta_{ij} - (c^2/g^{00})\psi_i^{(0)}\psi_j^{(0)}$, where $\psi_i^{(0)}$ is a basis element in the kernel of the matrix \mathbf{B} , $\hat{R}\psi_i^{(0)} = \mathbf{B}\psi_i^{(0)} = 0$, and $g^{00} = \sum_i \psi_i^{(0)}\psi_i^{(0)}$ is the corresponding element of the metric in flat coordinates. One can readily verify this identity by applying \mathbf{B}^2 to $\psi_i^{(\alpha)}$ with regard to the form (38) of the metric in flat coordinates. Using the expansion (39) once more, we obtain

$$\sum_{m \neq i, k} (r^m)^2 \beta_{mi} \beta_{mk} + (r^i)^2 \partial_i \beta_{ik} + (r^k)^2 \partial_k \beta_{ki} + r^i \beta_{ik} + r^k \beta_{ki} = 4c^2 H_i^{(0)} H_k^{(0)},$$

since $\psi_i^{(0)} = 2H_i^{(0)}$ (cf. (27)). This just means that there exists a third (this time, nonlocal) Hamiltonian structure determined by the metric $g_{(3)}^{ii} = (r^i)^2 g_{(1)}^{ii}$ of constant curvature c , since the expressions (13) computed for $g_{(3)}^{ii}$ read

$$\frac{1}{r^i r^k} [\mathbf{B}^2]_{ik} = \frac{4c^2}{r^i r^k} H_i^{(0)} H_k^{(0)}, \quad i \neq k.$$

The existence of the corresponding Hamiltonian for this nonlocal Hamiltonian structure readily follows from the results in [5].

Conversely, if an Egorov diagonal system with two flat metrics $g_{(1)}^{ii}$ and $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$ and with homogeneous functions β_{ik} has a third nonlocal Hamiltonian structure with constant curvature metric $g_{(3)}^{ii} = (r^i)^2 g_{(1)}^{ii}$, then $[\mathbf{B}^2]_{ij}/(r^i r^k) = d_i(r) \delta_{ij} - c \tilde{H}_i \tilde{H}_j / (r^i r^k)$, where $\tilde{H}_i(r) = 1/\sqrt{g_{(1)}^{ii}}$ and the diagonal entries $d_i(r)$ are some functions. Since the coefficients β_{ik} are homogeneous and invariant with respect to the shift $r^i \rightarrow r^i + \text{const}$, it follows that all products $\tilde{H}_i \tilde{H}_j$ are homogeneous of degree 0 and invariant with respect to the shift. Consequently, expressing the functions \tilde{H}_i themselves via these products, we find that the metric $g_{(1)}^{ii}$ is homogeneous of degree 0 and depends only on the differences of the Riemann invariants r^i . Applying the matrix \mathbf{B}^2 to any eigenvector or root vector, we readily find that all nonzero eigenvalues are the same and that there are no root vectors. ■

Theorem 10. *A homogeneous Egorov diagonal hydrodynamic type system (2) with flat metric $g_{(1)}^{ii}$ and second flat metric $g_{(2)}^{ii} = r^i g_{(1)}^{ii}$ has a third nonlocal Hamiltonian operator of the general form (11) with metric $g_{(3)}^{ii} = (r^i)^2 g_{(1)}^{ii}$ provided that the matrix \mathbf{B} is semisimple.*

Proof. In this general case, one has

$$[\mathbf{B}^2]_{ij} = c_1^2 \delta_{ij} - \sum_s c_1^2 \psi_i^{(0,s)} \psi_j^{(0,s)} + \sum_{\alpha \neq 1} (c_\alpha^2 - c_1^2) (\psi_i^{(\alpha)} \psi_j^{(-\alpha)} + \psi_i^{(-\alpha)} \psi_j^{(\alpha)}),$$

where the annihilators $\psi_i^{(0,s)}$ are basis elements of the kernel of the matrix \mathbf{B} , the $\psi_i^{(\alpha)}$ are its eigenvectors, and the $\psi_i^{(-\alpha)}$ are its eigenvectors with opposite eigenvalues arranged in a way such that the metric (38) is diagonal for $\psi_i^{(0,s)}$ and antidiagonal for $\psi_i^{(\alpha)}, \psi_i^{(-\alpha)}$, i.e., $\sum_i \psi_i^{(\alpha)} \psi_i^{(-\alpha)} = 1$; here we have chosen one of the eigenvalues c_1 as the “main” diagonal entry of the matrix \mathbf{B}^2 . This relation, just as in the preceding cases, can readily be verified by a straightforward substitution of the eigenvectors of \mathbf{B} . Thus, we can use (27) and write out the corresponding expansion (13) guaranteeing the existence of the Hamiltonian operator

(11). If \mathbf{B} has the eigenvalues $\pm 1/2$ (see Secs. 5 and 6 below), one should take $c_1 = -1/2$ to express the desired annihilators $\psi_i^{(\alpha)}$ via the corresponding $H_i^{(\alpha)}$ with the help of (27). ■

Note that $H_i^{(0)} = \psi_i^{(0)}/2$ can be found explicitly without quadratures even in the case of an arbitrary metric with homogeneous β_{ik} . For example, for $N = 3$ we obtain $\psi_i^{(0)} = r^j \beta_{kj} - r^k \beta_{jk}$, and for $N = 5$ one has $\psi_i^{(0)} = a_{jk} a_{lm} + a_{jl} a_{mk} + a_{jm} a_{kl}$, where $a_{jk} = r^j \beta_{kj} - r^k \beta_{jk}$. If the metric has the Egorov property, then we obtain $\psi_i^{(0)} = (r^j - r^k) \beta_{jk}$ for $N = 3$ and $\psi_i^{(0)} = (r^j - r^k)(r^l - r^m) \beta_{jk} \beta_{lm} + (r^j - r^l)(r^m - r^k) \beta_{jl} \beta_{mk} + (r^j - r^m)(r^k - r^l) \beta_{jm} \beta_{kl}$ for $N = 5$. In a similar way, one writes out $\psi_i^{(0)}$ for $N = 2n - 1 > 5$.

In the subsequent sections, we show how to establish the existence of second and third Hamiltonian structures for hydrodynamic type systems that are not written in Riemann invariants.

5. Averaged N -Phase Solutions of the Korteweg–de Vries Equation and the Nonlinear Schrödinger Equation

N -phase solutions of the KdV equation were averaged by the Whitham method in [22]. However, the Hamiltonian formalism for the averaged equations was developed later in [2, 15, 1]. The hydrodynamic type systems obtained by averaging inherit Galilean invariance, Hamiltonian property, and homogeneity. Moreover, since the KdV equation is Galilean invariant and hence has a pair of local Hamiltonian structures and a pair of conservation laws (see Theorem 1, Eq. (21))

$$\partial_t u = \partial_x [u^2 + \varepsilon^2 u_{xx}], \quad \partial_t [u^2 + \varepsilon^2 u_{xx}] = \partial_x [\frac{4}{3} u^3 - 3\varepsilon^2 u_x^2 + 2\varepsilon^2 (u^2)_{xx} + \varepsilon^4 u_{xxxx}],$$

it follows that after the averaging on an N -phase solution the resulting $(2N + 1)$ -component hydrodynamic type system is also Galilean invariant and has an Egorov metric (by Theorem 1) that is homogeneous and depends only on the differences of Riemann invariants. Hence, it has a pair of local Hamiltonian structures. The annihilators of the first local Hamiltonian structure of the averaged KdV equation comprise the averaged annihilator of the first local Hamiltonian structure of the KdV equation itself as well as N wave numbers (corresponding to the N phases $\theta_i = k_i x - \omega_i t$ of the quasiperiodic solution) and N corresponding partial derivatives of the averaged Lagrangian with respect to these wave numbers. A straightforward computation shows that the homogeneity degree of the functions $\psi_i^{(0)}$ corresponding to the averaged annihilator of the first local Hamiltonian structure is 0, the homogeneity degree of the functions $\psi_i^{(k)}$, $k = 1, \dots, N$, corresponding to the wave numbers is $-1/2$, and the homogeneity degree of the functions $\psi_i^{(N+k)}$, $k = 1, \dots, N$, corresponding to the partial derivatives of the averaged Lagrangian with respect to these wave numbers is $1/2$. Thus (by Theorem 9) the averaged KdV equations have a third nonlocal Hamiltonian structure associated with a constant curvature metric (see [25]). (The existence of the first two local Hamiltonian structures was established earlier in [2].)

Multiphase solutions of the nonlinear Schrödinger equation were averaged in the diploma paper of the first author. Later, this result was published in [6]. It was obtained independently in [21]. It was proved that averaged N -phase solutions of the NLS equation have three local Hamiltonian structures and a fourth, nonlocal Hamiltonian structure defined by a differential-geometric Poisson bracket with constant curvature metric. The canonical form of the first three Hamiltonian structures in flat coordinates was given in [1] and [15]. The nonlinear Schrödinger equation written out in the Hasimoto form is Galilean invariant and

has three local Hamiltonian structures and the pair of conservation laws

$$\partial_t |u|^2 = i\partial_x [u\bar{u}_x - \bar{u}u_x], \quad i\partial_t [u\bar{u}_x - \bar{u}u_x] = \partial_x [|u|^4 + 4|u_x|^2 - (|u|^2)_{xx}].$$

Reproducing the above argument for the KdV equation in the case of the nonlinear Schrödinger equation, one can also show that these properties are inherited under N -phase averaging (see [8]), which is a good illustration of Theorems 1 and 8.

6. The Benney–Zakharov and Yajima–Oikawa–Mel’nikov Systems (the Dispersionless Limit)

The Benney system (Zakharov’s reduction; see [16, 4])

$$u_t^k = \partial_x \left[\frac{(u^k)^2}{2} + \sum_{m=1}^N \eta^m \right], \quad \eta_t^k = \partial_x (u^k \eta^k) \quad (40)$$

is Galilean invariant ($u^k \rightarrow u^k + c$, $x \rightarrow x - ct$) and has a pair of conservation laws of the form (see Theorem 1, Eq. (21))

$$\begin{aligned} \partial_t \left(\sum_{m=1}^N \eta^m \right) &= \partial_x \left(\sum_{m=1}^N u^m \eta^m \right), \\ \partial_t \left(\sum_{m=1}^N u^m \eta^m \right) &= \partial_x \left[\sum_{m=1}^N (u^m)^2 \eta^m + \frac{1}{2} \left(\sum_{m=1}^N \eta^m \right)^2 \right] \end{aligned}$$

and the first local Hamiltonian structure

$$u_t^k = \partial_x \frac{\delta H}{\delta \eta^k}, \quad \eta_t^k = \partial_x \frac{\delta H}{\delta u^k}$$

determined by the following homogeneous Egorov metric depending on the differences of Riemann invariants (in the Riemann invariants λ^k , $k = 1, \dots, 2N$; see [13]):

$$g^{ii} = \sum_{m=1}^N \frac{\eta^m}{(\mu^i + u^m)^3},$$

where the characteristic velocities μ^i and the Riemann invariants λ^i can be found from the system (see [23])

$$\lambda^i = \mu^i + \sum_{m=1}^N \frac{\eta^m}{\mu^i + u^m}, \quad 1 = \sum_{m=1}^N \frac{\eta^m}{(\mu^i + u^m)^2}.$$

Comparing the homogeneity degrees of the flat coordinates (u^k, η^k) and the metric g^{ii} , we see that the homogeneity degree of the functions $\psi_i^{(k)}$, $k = 1, \dots, N$, corresponding to the first half of flat coordinates u^k is $-1/2$ and the homogeneity degree of the functions $\psi_i^{(N+k)}$, $k = 1, \dots, N$, corresponding to the second half of flat coordinates η^k is $1/2$. It follows by Theorems 1 and 8 that the Benney–Zakharov solution (40) has also second and third local Hamiltonian structures (see [10, 9]).

In the dispersionless limit, the Yajima–Oikawa system (known as the long-short resonance) generalized to the N -component case (Mel’nikov’s system I; see [26]) has the form

$$u_t^k = \partial_x \left[\frac{(u^k)^2}{2} + w \right], \quad \eta_t^k = \partial_x (u^k \eta^k), \quad w_t = \partial_x \left[\sum_{m=1}^N \eta^m \right].$$

It is also Galilean invariant ($w \rightarrow w + c$, $x \rightarrow x - ct$) and has a pair of conservation laws of the form

$$\partial_t w = \partial_x \left(\sum_{m=1}^N \eta^m \right), \quad \partial_t \left(\sum_{m=1}^N \eta^m \right) = \partial_x \left[\sum_{m=1}^N u^m \eta^m \right]$$

prescribed in Theorem 1 and the first local Hamiltonian structure

$$u_t^k = \partial_x \frac{\delta H}{\delta \eta^k}, \quad \eta_t^k = \partial_x \frac{\delta H}{\delta u^k}, \quad w_t = \partial_x \frac{\delta H}{\delta w}$$

determined by the following homogeneous Egorov metric depending on the differences of Riemann invariants (in the Riemann invariants λ^k , $k = 1, \dots, 2N + 1$):

$$g^{ii} = 1 - 2 \sum_{m=1}^N \frac{\eta^m}{(\mu_i + u^m)^3},$$

where the characteristic velocities μ_i and the Riemann invariants λ_i can be found from the system

$$\lambda^i = \frac{\mu_i^2}{2} + w - \sum_{m=1}^N \frac{\eta^m}{\mu_i + u^m}, \quad \mu_i + \sum_{m=1}^N \frac{\eta^m}{(\mu_i + u^m)^2} = 0.$$

Comparing the homogeneity degrees of the flat coordinates (w, u^k, η^k) and the metric g^{ii} we see that the homogeneity degree of the function $\psi_i^{(0)}$ corresponding to the flat coordinate w is zero, the homogeneity degree of the functions $\psi_i^{(k)}$, $k = 1, \dots, N$, corresponding to the first half of flat coordinates u^k is $-1/2$, and the homogeneity degree of the functions $\psi_i^{(N+k)}$, $k = 1, \dots, N$, corresponding to the second half of flat coordinates η^k is $1/2$. It follows by Theorems 1 and 9 that the Yajima–Oikawa–Mel’nikov system has also second (local) and third (nonlocal, with constant curvature metric) Hamiltonian structures.

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